§6.2-6.4. (Natural) Logarithm/Exponential Functions and their Applications.

- log $\boxed{y=\log_a x}$ $\boxed{y=a^x}$ exponential

  nature log $\boxed{y=\ln x}$ $\boxed{y=e^x}$ natural exponential.

- Motivation: Generalization of integer power and its reverse

  2^n: $2^3 = 2\times2\times2 = 8$, $2^4 = 2\times2\times2\times2 = 16$ $\Rightarrow 2^{3.5}$? $\xrightarrow{\text{exp}}$ 2^x for any x.

  Reverse (Inverse)

  $3 \longmapsto 8$, $4 \xmapsto{\text{exp}} 16$ reverse: $8 \longmapsto 3$, $16 \xmapsto{\text{log}} 4$ $\log_2 x$.

  $2 \longmapsto a$ $(a>0)$. $\boxed{y=a^x}$. Special case: $a=e$, $\boxed{y=e^x}$.

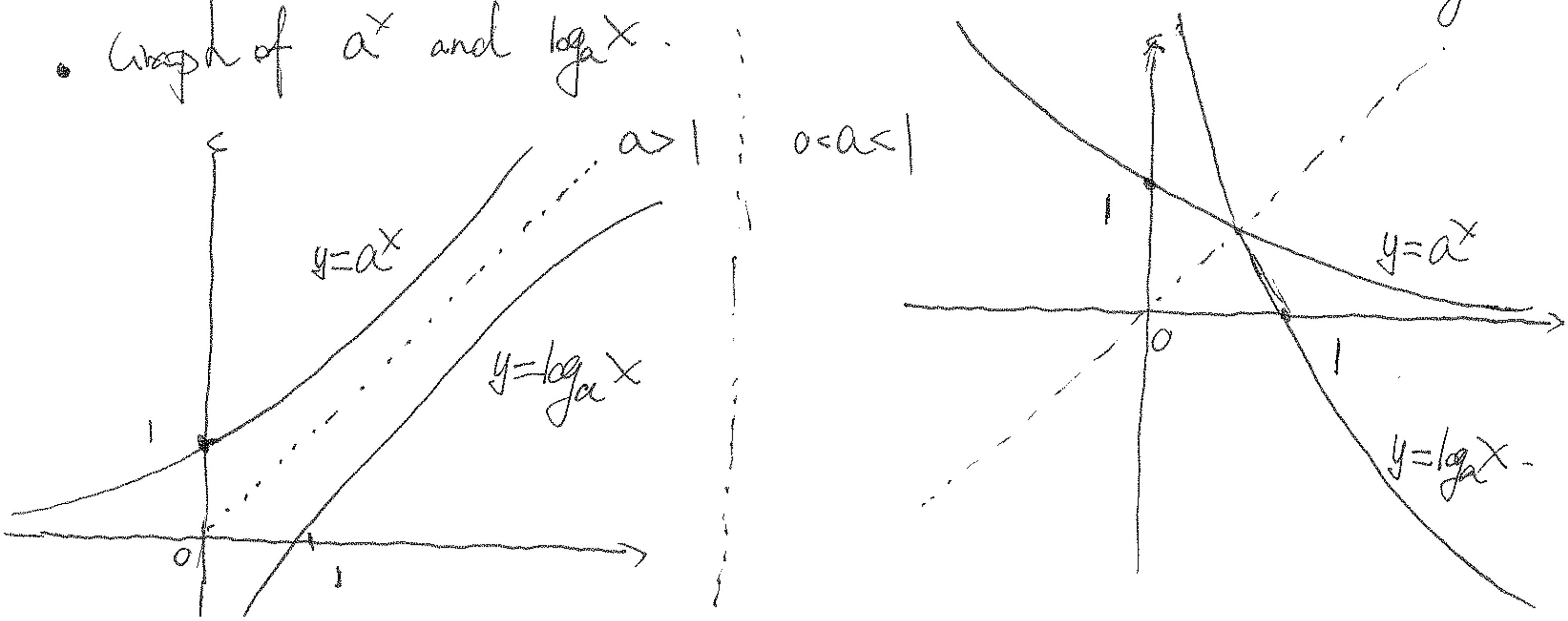
  exp-function with base a. natural exp.

  Inverse

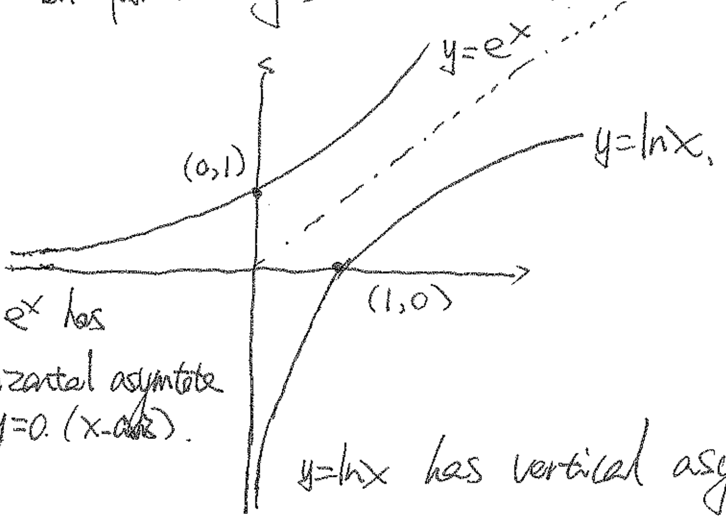
  $\boxed{y=\log_a x}$ Special case: $a=e$, $\boxed{y=\ln x}$

  log-function with base a natural log.

- Graph of a^x and $\log_a x$.



In particularly, $a = e > 1$



$y = e^x$ goes to $+\infty$ as $x \to +\infty$

$y = \ln x$. goes to $+\infty$ as $x \to +\infty$.

$y = e^x$ y-intercept $(0,1)$

$y = \ln x$ x-intercept $(1,0)$.

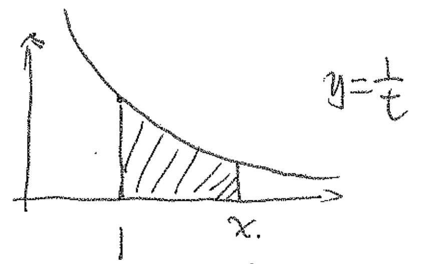
$y = e^x$ has horizontal asymptote $y=0$. (x-axis).

$y = \ln x$ has vertical asymptote $x=0$ (y-axis) from right of $x=0$.

§6.2. Natural log.

- A former way to define $y = \ln x$.

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt \quad x > 0}$$



$$\Rightarrow \boxed{(\ln x)' = \frac{1}{x}.}$$

The area below $\frac{1}{t}$ from 1 to x. is a function of x and is denoted by $\ln x$.

(Hint: Fundamental Thm of Calculus, part I.)

- Properties of ln:

① $\ln(x \cdot y) = \ln x + \ln y$

② $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

③ $\ln(x^r) = r \cdot \ln x$.

Q: Find $\ln(x^a \cdot y^b)$

$= a \cdot \ln x + b \cdot \ln y$

④ $\lim_{x \to 0^+} \ln x = -\infty$, $\lim_{x \to +\infty} \ln x = +\infty$

⑤ $\frac{d}{dx} \ln|x| = \frac{1}{x}$, $\int \frac{1}{x}\,dx = \ln|x| + C$.

e.g. 1 Simply the following expression (express it as a single logarithm)

$$\frac{1}{3}\cdot\ln(x+2)^3 + \frac{1}{2}\cdot\left[\ln x - \ln(x^2+3x+2)^2\right]$$

$$= \frac{1}{3}\cdot 3\ln(x+2) + \frac{1}{2}\cdot\ln x - \frac{1}{2}\cdot 2\cdot\ln(x^2+3x+2)$$

$$= \ln(x+2) + \frac{1}{2}\ln x - \ln(x+1)\cdot(x+2)$$

$$= \ln(x+2) + \frac{1}{2}\ln x - \left[\ln(x+1) + \ln(x+2)\right]$$

$$= \frac{1}{2}\ln x - \ln(x+1) = \ln x^{\frac{1}{2}} - \ln(x+1) = \boxed{\ln\frac{x^{\frac{1}{2}}}{x+1}}$$

Problems related to ~~differ~~ derivative, limit and integral.

- Derivatives: $\begin{cases} \text{chain rule : } [f(g(x))]' = f'(g(x))\cdot g'(x), \\ \star\ \boxed{\text{logarithmic differentiation}} \end{cases}$

e.g. 2: Compute $\frac{d}{dx}\ln\frac{3x+1}{\sqrt{x-2}}$.

step 1: Simply the log via algebra property.

$$\ln\frac{3x+1}{\sqrt{x-2}} = \ln(3x+1) - \ln\sqrt{x-2} = \ln(3x+1) - \frac{1}{2}\ln(x-2)$$

step 2: Take the derivative:

$$\frac{d}{dx}\ln\frac{3x+1}{\sqrt{x-2}} = \frac{d}{dx}\ln(3x+1) - \frac{d}{dx}\frac{1}{2}\ln(x-2)$$

(chain rule): $= \boxed{\frac{1}{3x+1}\cdot 3 - \frac{1}{2}\cdot\frac{1}{x-2}}$

eg 3 Compute y' where $y = \dfrac{x\cdot\sqrt{x^2+1}}{(3x+2)^5}$

Hint: Derivate of a sum is much easier than derivative of a product.

log (or ln) can turn a ~~sum~~ product into a sum.

Step 1: $\ln y = \ln \dfrac{x\cdot\sqrt{x^2+1}}{(3x+2)^5} = \ln x\cdot\sqrt{x^2+1} - \ln(3x+2)^5$

$= \ln x + \ln\sqrt{x^2+1} - \ln(3x+2)^5$

$= \ln x + \frac{1}{2}\cdot\ln(x^2+1) - 5\cdot\ln(3x+2) \cdots\cdots *$

Step 2: Take derivate w.r.t. x in the above equation.

L.H.S: $\frac{d}{dx}\ln y = \frac{1}{y}\cdot y'$ (chain rule. Caution: $y=y(x)$ is a function of x).

R.H.S: $\frac{d}{dx} * = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\cdot\frac{1}{3x+2}\cdot 3$

i.e. $\frac{1}{y}\cdot y' = \frac{1}{x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$

Step 3: multiply by y both sides, solve for y'.

$y' = y\cdot\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$ ← Caution: Not Done Yet. Need to replace y by the expression of x terms.

$$\boxed{y' = \frac{x\cdot\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]}$$

##

- u-Sub related to ln.

e.g 4. $\int \frac{x^3}{x^4+2}\,dx$.

u-Sub Setting:
$u = x^4+2$, $du = 4x^3 \cdot dx$.
$\frac{1}{4}du = x^3 \cdot dx$.

$$\overset{\text{u-Sub}}{=} \int \frac{1}{x^4+2} \cdot x^3 dx$$
$$= \int \frac{1}{u} \cdot \frac{1}{4} du$$
$$= \frac{1}{4} \int \frac{1}{u} \cdot du = \frac{1}{4} \cdot \ln|u| + C \overset{\text{Back to X.}}{=\!=} \boxed{\frac{1}{4} \cdot \ln|x^4+2| + C}$$

☆☆ eg.5. (webwork) Evaluate $\int_1^{e^{\frac{\pi}{6}}} \frac{\tan(\ln x^2)}{3x} \cdot dx$.

u-Sub:
$u = \ln x^2$ ($= 2\ln x$)
$du = 2 \cdot \frac{1}{x} dx$.
$\frac{1}{2} du = \frac{1}{x} \cdot dx$.

$x = e^{\frac{\pi}{6}} \xrightarrow{u=\ln x^2} u = \ln e^{(\frac{\pi}{6})^2} = \ln e^{\frac{\pi}{3}} = \frac{\pi}{3}$
$x = 1$ $\quad u = \ln 1 = 0$

$$\overset{\text{u-Sub}}{=} \int_0^{\frac{\pi}{3}} \frac{\tan u}{3} \cdot \frac{1}{2} \cdot du$$
$$= \frac{1}{6} \int_0^{\frac{\pi}{3}} \tan u \cdot du$$
$$= \frac{1}{6} \ln|\sec u| \Big|_0^{\frac{\pi}{3}}$$
$$= \frac{1}{6} \ln \sec\frac{\pi}{3} - \frac{1}{6} \ln \sec 0$$
$$= \frac{1}{6} \cdot \ln 2 - \frac{1}{6} \cdot \ln 1$$
$$= \boxed{\frac{1}{6} \cdot \ln 2}$$

Remark: ① $(e^{\frac{\pi}{6}})^2 = e^{\frac{\pi}{6} \times 2}$ will be discussed in §6.3

② $\ln e = 1$.
$\ln 1 = 0$

③ $\int \tan x\, dx$
$= \ln|\sec x| + C$
will be given in the exam formula sheet.
See proof in ex 13 in the text book Page 426

④ $\sec x = \frac{1}{\cos x}$
$\sec\frac{\pi}{3} = 2$
$\sec 0 = 1$

§6.3. Natural exp. $y=e^x$ Domain: $(-\infty,+\infty)$. Range: $(0,+\infty)$.

is the inverse function of $y=\ln x$.

- Algebra properties: $e^0=1$, $e^{-\infty}=0$, $e^{+\infty}=+\infty$, $\ln e=1$.

$$e^{x+y}=e^x\cdot e^y \;;\quad e^{x-y}=\frac{e^x}{e^y}\;;\quad (e^x)^r=e^{r\cdot x}$$

- Relation with ln: $e^{\ln x}=x$; $\ln e^x=x$.

$$\boxed{y=e^x \iff \ln y=x} \qquad \boxed{y=\ln x \iff e^y=x}$$

- ~~Differentiation~~:

$$\frac{d}{dx}e^x=(e^x)'=e^x \iff \int e^x\,dx=e^x+C.$$

- Remark: Linear Integration Rule for $\ln x$ and e^x.

$$☆\ \boxed{\int e^{ax+b}\,dx=\frac{1}{a}\cdot e^{ax+b}+C\;;\quad \int\frac{1}{ax+b}\,dx=\frac{1}{a}\ln|ax+b|+C}$$

(Prove the above two equations via u-sub: $u=ax+b$).

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e.g. 6: Find $y'(t)$ of $y=\ln(5t\cdot e^{-t})$.

Solution 1: (Compute Directly)

$$y'=\underbrace{\frac{1}{5t\cdot e^{-t}}\cdot(5t\cdot e^{-t})'}_{\text{chain rule}}\underset{\text{product rule}}{=}\frac{1}{5t\cdot e^{-t}}\cdot\left[(5t)'\cdot e^{-t}+5t\cdot(e^{-t})'\right]$$

↓ chain rule

$$=\frac{1}{5t\cdot e^{-t}}\cdot\left[5\cdot e^{-t}+5t\cdot(e^{-t})\cdot(-1)\right]$$

$$\text{(Simplify)}\quad=\boxed{\frac{1}{t}-1}.$$

Solution 2: Simplify first.

$$y=\ln(5t\cdot e^{-t})=\ln 5+\ln t+\ln e^{-t}=\ln 5+\ln t+(-t)$$

$$\Rightarrow y'=(\ln 5)'+(\ln t)'+(-t)'=\boxed{0+\frac{1}{t}-1}$$

☆☆: e.g. 7 (FTC I)

Compute $y'(x)$ if $y = \int_{10^5}^{\ln(2x)} \cos(5e^t)\cdot dt$

Hint: FTC I chain rule formula: $\frac{d}{dx}\int_a^{u(x)} \cdot f(t)dt = f(u(x)) \cdot u'(x)$.

SLN: $y'(x) = \frac{d}{dx}\int_{10^5}^{\ln(2x)} \cos(5e^t)\cdot dt$

$= \cos(5\cdot e^{\ln(2x)}) \cdot (\ln(2x))'$

replace t by the function in the upper limit and times its derivative.

(Simplify) $= \cos(5\cdot 2x)\cdot \frac{1}{2x}\cdot 2 = \boxed{\cos(10x)\cdot \frac{1}{x}}$ ※

☆ eg 8. (Implicit Diff rule)

Find $\frac{dy}{dx}$ in terms of x and y if $\ln y = y^2 \cdot e^{3x}$

$(\ln y)' = (y^2\cdot e^{3x})' \Rightarrow \frac{1}{y}\cdot y' = (y^2)'\cdot e^{3x} + y^2\cdot (e^{3x})'$

$\frac{1}{y}\cdot y' = 2y\cdot y'\cdot e^{3x} + y^2\cdot e^{3x}\cdot 3$

Solve for y' $\left(\frac{1}{y} - 2y\cdot e^{3x}\right)\cdot y' = y^2\cdot e^{3x}\cdot 3$

$\Rightarrow \boxed{y' = \frac{y^2\cdot e^{3x}\cdot 3}{\frac{1}{y} - 2y\cdot e^{3x}}}$ ※

eg 9. (u-Sub rule for e^x).

$\int 2x\cdot e^{x^2}dx$ $u = x^2$, $du = 2x\,dx$

$= \int e^u\cdot du = e^u + C = \boxed{e^{x^2} + C}$.

§6.4 General log/exp.

$e^x \Rightarrow a^x$, $\ln x \Rightarrow \log_a x$ ($\ln x$ is actually short for $\log_e x$)

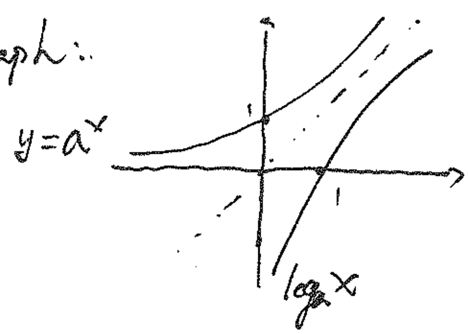
- $y=a^x$ and $y=\log_a x$ are inverse functions to each other.

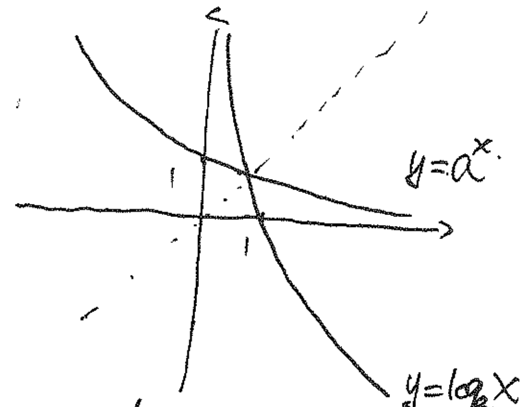
Domain $(-\infty,+\infty)$ $(0,+\infty)$

Range $(0,+\infty)$ $(-\infty,+\infty)$.

- Graph:

$a>1$ | $a<1$





Remark: Sometimes it is convinient to imagine $a=2$ and $\frac{1}{2}$. for the above two graphs.

- Properties: $\boxed{a^x = e^{x(\ln a)} \qquad \log_a x = \frac{\ln x}{\ln a}}$ $\ln a$ is a constant.

e.g. $7^x = e^{(\ln 7)\cdot x}$

$\log_{\sqrt{2}} x = \frac{\ln x}{\ln\sqrt{2}}$.

$$\boxed{\begin{aligned}(a^x)' &= a^x\cdot\ln a\\ \int a^x\,dx &= \frac{1}{\ln a}\cdot a^x + C\end{aligned}} \qquad (\log_a x)' = \frac{1}{x\ln a}$$

$$a^{x+y} = a^x\cdot a^y, \quad a^{x-y} = \frac{a^x}{a^y}, \quad \boxed{(a^x)^y = e^{x\cdot y}, \quad (ab)^x = a^x\cdot b^x}$$

eg.10. $\log_3 81 = \frac{\ln 81}{\ln 3} = \frac{\ln 3^4}{\ln 3} = \frac{4\cdot\ln 3}{\ln 3} = 4$; $\log_{25} a2 = \frac{\ln a2}{\ln 25} = \frac{\ln 5^{-1}}{\ln 5^2} = \boxed{\frac{-1}{2}}$.

☆ eg.11. $\int \left(\frac{1}{7}\right)^{\tan x}\cdot\sec^2 x\cdot dx$. u-sub: $u=\tan x$, $du = \sec^2 x\cdot dx$.

$$= \int\left(\frac{1}{7}\right)^u\cdot du$$

$$= \frac{1}{\ln\frac{1}{7}}\cdot\left(\frac{1}{7}\right)^u + C = \boxed{\frac{1}{\ln\frac{1}{7}}\cdot\left(\frac{1}{7}\right)^{\tan x} + C}$$

Remark: you can move one step further to write $\ln\frac{1}{7} = -\ln 7$

§6.5/9.3, Initial Value Problems

eg.1 (Motivation).

Give a function $y = y(t)$ satisfying:

① The rate of change of y is proportional to y with ratio k. (constant).

② The initial value of y is $y(0) = C$. (constant)

Find $y(t)$ (as a function of t).

sln: $\begin{cases} \dfrac{d\,y(t)}{dt} = k \cdot y(t) \\ y(0) = C. \end{cases}$ Notice that the unknown is not a variable but a function.

One can CHECK that $\boxed{y(t) = C \cdot e^{kt}}$ satisfies the above equation.

- General equations and Method to solve the eqns.

Separable eqns: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)} \Leftrightarrow \boxed{h(y) \cdot dy = g(x) \cdot dx}$.

eg.2. Rewrite $\sec x \cdot \dfrac{dy}{dx} = e^{3y + \sin x}$ in its separable form:

Hint: $\sec x = \dfrac{1}{\cos x}$, $e^{a+b} = e^a \cdot e^b$

sln: $\sec x \cdot \dfrac{dy}{dx} = e^{3y + \sin x} = e^{3y} \cdot e^{\sin x}$ Move all y terms to the left and all x terms to the right.

$\dfrac{1}{e^{3y}} \cdot dy = e^{\sin x} \cdot \dfrac{1}{\sec x} \cdot dx$. Hint: $\dfrac{1}{e^{3y}} = e^{-3y}$

$\Rightarrow \boxed{e^{-3y}\, dy = e^{\sin x} \cdot \cos x \, dx}$

- The solution to the separable equation: Integrate both sides

$$\boxed{\int h(y)\,dy = \int g(x)\,dx}$$

e.g.3. Solve the initial value problem. $y' = e^{-y}(2x-4)$, $y(5)=0$.

sln: S1: Rewrite in its sep form: $\frac{dy}{dx} = e^{-y}\cdot(2x-4)$

$$\Leftrightarrow e^{y}\cdot dy = (2x-4)\cdot dx.$$

S2: Integrate both sides:

L.H.S: $\int e^{y}\,dy = e^{y}$; R.H.S: $\int 2x-4\,dx = x^2-4x$.

Set them equal and $\boxed{\text{ADD A CONSTANT C}}$ on one side

$e^{y} = x^2-4x+C$

S3: Use initial condition to ~~solve~~ find the unknown C.

$y(5)=0 \Rightarrow x=5$ & $y=0 \Rightarrow e^0 = 5^2-4\cdot5+C \Rightarrow 1 = 5+C$

$\Rightarrow \boxed{C=-4}$

S4: Solve for $y=y(x)$.

$e^{y} = x^2-4x-4 \xrightarrow{\text{take ln}} \boxed{y=\ln|x^2-4x-4|}$.

e.g 4. Solve the equation in e.g. 2 with $y(0)=0$.

$\int e^{-3y}\,dy = \int e^{\sin x}\cdot\cos x\,dx$

$\frac{1}{-3}\cdot e^{-3y} = e^{\sin x} + C$.

$\int e^{-3y}\,dy \xrightarrow[du=-3dy]{u=-3y} \int e^{u}\cdot\frac{du}{-3} = -\frac{1}{3}e^{u} = \boxed{-\frac{1}{3}e^{-3y}}$

$\int e^{\sin x}\cdot\cos x\,dx \xrightarrow[du=\cos x\cdot dx]{u=\sin x} \int e^{u}\,du = e^{u} = \boxed{e^{\sin x}}$

$x=0, y=0.\ \sin 0 = 0,\ e^0=1$

$-\frac{1}{3}\cdot e^0 = e^0 + C \Rightarrow -\frac{1}{3} = 1+C \Rightarrow C = -\frac{4}{3}$

$-\frac{1}{3}\cdot e^{-3y} = e^{\sin x} - \frac{4}{3} \Rightarrow e^{-3y} = -3\cdot e^{\sin x} + 4$

$\Rightarrow -3y = \ln|-3e^{\sin x}+4| \Rightarrow \boxed{y = -\frac{1}{3}\ln|-3e^{\sin x}+4|}$